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Onset condition of modulated Rayleigh-Bénard convection at low frequency

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The onset condition of convection in a layer of fluid bounded by isothermal walls, with lower temperature varying sinusoidally in time at very low nondimensional modulation frequency, is derived in closed form, based on the Floquet theory and using a matched-asymptotic WKB method.

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It is well known that a layer of fluid develops convection rolls when the basic temperature gradient is sufficiently large. For isothermal walls the onset condition [1] corresponds to critical Rayleigh number $\mathrm{Ra}_c = 1707.762$ and critical wavelength of the rolls $2\pi d/3.117$, where d is the layer thickness. Here, the lower wall temperature is allowed to vary very slowly in time in a sinusoidal fashion, whereas the upper wall temperature is held at the mean low-wall temperature. In this situation convection occurs during the supercritical part of the cycle, but damps out in the subcritical part of the cycle [2]. During the supercritical part of the cycle, the amplitude of convection can grow to a significantly large value on a linear basis if the nondimensional modulation frequency ω, defined by $\omega = \omega^* d^2/\kappa$, is small. The parameter ω measures the ratio of the dimensional frequency ω^* and the thermal diffusive rate κ/d^2, where κ is the fluid thermal diffusivity. In reality, nonlinear effects set in to limit amplitude growth. During the growth phase random forcings can couple to nonlinear effects and break the symmetry of the Floquet cycle. Transitions to other type of flow patterns can occur. Such possibility has been considered for the Taylor vortex flow [3], and and typically a strong forcing is required.

For small background noises, a sufficient condition for instability is the occurrence of net growth over a complete modulation cycle, i.e., in the Floquet sense. So far, no explicit stability condition has been derived for modulated Rayleigh-Bénard (RB) convection in the limit $\omega \to 0$. As $\omega \to 0$, most direct numerical methods yield ill-conditioned solutions. For instance, for the onset problem Rosenblat and Herbert [4], Rosenblat and Tanaka [5], Yih and Li [6], and Or and Kelly [7] all computed the stability limit down to a certain ω value and stopped. Dowden [8] used a WKB method but without a matched asymptotic approach. Because of that the onset condition derived is valid only for a single frequency value, $\omega = 0$. Considerable nonlinear analyses have been done, for instance, using the amplitude equation approach [9] or modal method involving a nonlinear Mathieu equation [10]. However, the limit $\omega \to 0$ was not examined in the analyses.

The onset condition is derived here for the asymptotic range, using a matched-asymptotic WKB method [11]. Several simplifications make the problem tractable analytically. First, as $\omega \to 0$ the basic temperature gradient is constant. In the leading order, the stability equations do not have explicit dependence on the vertical coordinate. Second, a stress-free wall condition is assumed. With this condition the Mathieu equation is exact for the asymptotic stability problem. For the present problem the coefficients of the Mathieu equation depends inversely on ω and ω^2, and the solution in the limit $\omega \to 0$ will be obtained by a matched asymptotic method.

Consider an infinite layer with upper and lower wall temperatures maintained at T_0^* and $T_0^* + T_\delta^* \cos \omega^* t^*$. The length, time, velocity, and temperature are scaled, respectively, by d, ω^{*-1}, and κ/d. The nondimensional perturbation equations subjected to stress-free wall conditions are

$$\omega \mathrm{Pr}^{-1} \nabla^2 w_t - \nabla^4 w = \mathrm{Ra} \nabla_\perp^2 \theta, \tag{1}$$

$$\omega\, \theta_t - \nabla^2 \theta = -T_z(z,t) w, \tag{2}$$

$$w(x,y,0,t) = w_{zz}(x,y,0,t) = \theta(x,y,0,t) = 0, \tag{3}$$

and

$$w(x,y,1,t) = w_{zz}(x,y,1,t) = \theta(x,y,1,t) = 0,$$

where $w(x,y,z,t)$ and $\theta(x,y,z,t)$ are, respectively, the perturbational vertical velocity and temperature and $T_z(z,t)$ is the basic temperature gradient. Here the subscript variables of t and z denote partial derivatives with respect to time and the vertical coordinate. The leading order of T_z is z-indpendent, as $T_z \to -\{1 + 0(\omega^{1/2}) \phi(z,t)\} \cos t$ when $\omega \ll 1$, such that $\phi(z,t)$ is an $O(1)$ function. The Rayleigh number and Prandtl number are, respectively, defined as $\mathrm{Ra} = g T_\delta^* d^3/\kappa\nu$ and $\mathrm{Pr} = \nu/\kappa$.

Consider a solution of the form

$$w = e^{-(1+\mathrm{Pr})t/2\epsilon} W(t) \sin(\pi z) \cos(k_x x + k_y y), \tag{4}$$

$$\theta = \Delta^{-1} e^{-(1+\mathrm{Pr})t/2\epsilon} \Theta(t) \sin(\pi z) \cos(k_x x + k_y y), \tag{5}$$

where k_x and k_y are the horizontal wave numbers and $\Delta = (\pi^2 + k^2)$ with $k^2 = (k_x^2 + k_y^2)$ and $\epsilon = \omega/\Delta$, with the coefficient W governed by the Mathieu equation

$$\epsilon^2 \ddot{W} = Q(t) W, \quad \text{with} \quad Q(t) = (\delta + r \cos t), \tag{6}$$

where $\delta = (1-\mathrm{Pr})^2/4$, $r = \mathrm{Pr Ra}/\mathrm{Ra}_n$, and $\mathrm{Ra}_n = \Delta^3/k^2$.

The zero-frequency asymptotic limit corresponds $\epsilon \to 0$. For onset of instability in the Floquet sense, $Q(t)$ changes sign in the period $[0,2\pi]$, and $t = t_{1,2}$ are the pair of turning points at which $Q = 0$. The character of the solution changes locally in the interval $[0,\pi]$ as the sign of $Q(t)$ changes. In order to obtain the solution of $W(t)$ over a complete Floquet

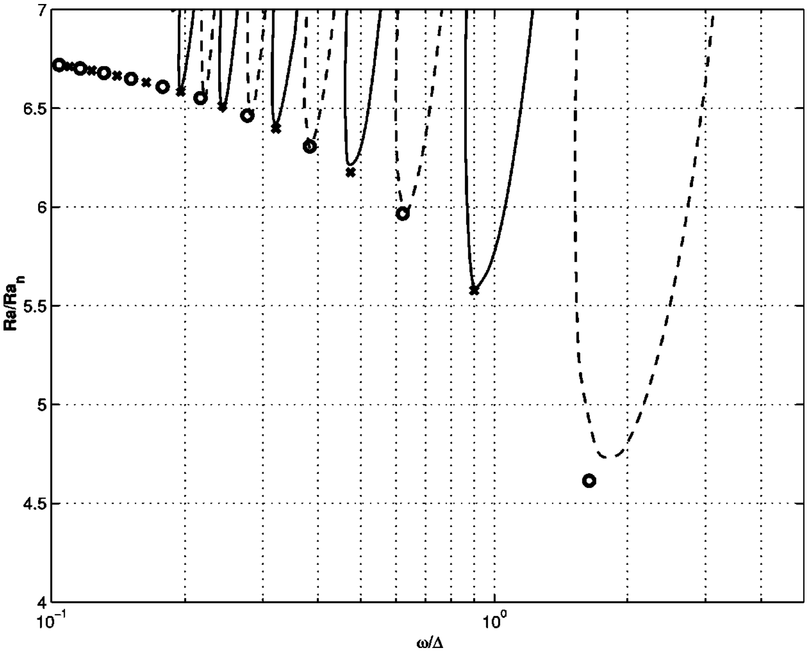

FIG. 1. Exact neutral curve and analytical onset condition.

period, several inner and outer solutions, together with their locally scaled equations, have to be constructed and matched asymptotically across the turning points [9]. Once $W(t)$ is obtained, the onset condition of convection is $|W(2\pi)/W(0)|>e^{(1+\mathrm{Pr})\pi/\epsilon}$. The procedure is straight forward but tedious and will not be described here. It yields the onset condition for the synchronous mode

$$\sin\left(\frac{1}{\epsilon}\Phi_2\right)+2\cos\left(\frac{1}{\epsilon}\Phi_2\right)e^{2/\epsilon\Phi_1}>e^{(1+\mathrm{Pr})\pi/\epsilon}, \tag{7}$$

and another onset condition for the subharmonic mode,

$$\sin\left(\frac{1}{\epsilon}\Phi_2\right)+2\cos\left(\frac{1}{\epsilon}\Phi_2\right)e^{2/\epsilon\Phi_1}<-e^{(1+\mathrm{Pr})\pi/\epsilon}, \tag{8}$$

where $\Phi_{1,2}$ are integrals given by

$$\Phi_1=\int_0^{t_1}Q^{1/2}d\bar{t},\quad \Phi_2=\int_{t_1}^{t_2}(-Q)^{1/2}d\bar{t}. \tag{9}$$

Note that in general both synchronous and subharmonic solutions are expected. For $\Phi_2=0$, i.e., $Q(t)$ does not change sign, Eqs. (7) and (8) give the more restricted onset condition

$$e^{1/\epsilon(2\Phi_1-(1+\mathrm{Pr})\pi)}\geqslant 1. \tag{10}$$

Equations (7) and (8) can be simplified further. For illustration, consider $\mathrm{Pr}=1$. In this case $\Phi_1=\sqrt{r}I_0$ and $\Phi_2=2\sqrt{r}I_0$, where $I_0=\int_0^{\pi/2}\sqrt{\cos t}dt\approx 1.1981$. From Eqs. (7) and (8), the neutral curve for the synchronous mode (plus sign) and subharmonic mode (minus sign) is given by

$$2\cos\left\{\frac{\sqrt{r}}{\epsilon}I_0\right\}=\pm e^{1/\epsilon(2\pi-2\sqrt{r}I_0)}. \tag{11}$$

In Fig. 1, the neutral curve is computed from the exact Oberbeck-Boussinesq equations down to $\omega/\Delta=0.2$. These are loops alternating in synchronous (solid) and subharmonic (dashed) modes. Of particular interest is the local minima of $r=\mathrm{Ra}/\mathrm{Ra}_n$. The left-hand side of Eq. (11) is a cosine function and therefore has a period of 2π. The minima of the synchronous and subharmonic modes correspond to the maxima ($\theta=2n\pi$) and minima ($\theta=2(n+1/2)\pi$) of the cosine function $\cos\theta$. For the synchronous mode, the minima are determined by

$$2\sqrt{r}I_0=\epsilon 2n\pi,\quad 2=e^{1/\epsilon(2\pi-2\sqrt{r}I_0)}. \tag{12}$$

These two simultaneous equations give the minima

$$\epsilon_n^S=\frac{2\pi}{\ln 2+2n\pi}, \tag{13}$$

$$r_n^S=\left(\frac{2n\pi}{\ln 2+2n\pi}\right)^2\left(\frac{\pi}{I_0}\right)^2,\quad n=1,2,3,\ldots. \tag{14}$$

The case $n=0$ is not admitted as solution because $r=0$ and ϵ becomes arbitrary. For the subharmonic mode, the minima are determined by

$$2\sqrt{r}I_0=\epsilon 2(n-1/2)\pi,\quad 2=e^{1/\epsilon(2\pi-2\sqrt{r}I_0)}. \tag{15}$$

These two simultaneous equations give the minima

$$\epsilon_n^H=\frac{2\pi}{\ln 2+2(n-1/2)\pi}, \tag{16}$$

$$r_n^H=\left(\frac{2(n-1/2)\pi}{\ln 2+2(n-1/2)\pi}\right)^2\left(\frac{\pi}{I_0}\right)^2, \tag{17}$$

where $n=1,2,3,\ldots$. In Fig. 1, the minima from Eqs. (13), (14), (16) and (17) are shown, respectively, by crosses and circles, for the range $\epsilon=0.1$ to 5.0. These points terminate to the right at $n=1$. The outermost loop of the exact solution corresponds to a subharmonic mode, with minimum at $\epsilon_1^H = 1.64$ and $r_1^H = 4.61$. Except for the outermost dashed loop, the crosses and circles match the exact minima of the loops very well.

As ϵ decreases toward zero, the neutral curve loops occur indefinitely in alternating fashion. The analytical result indicates $r_n^S, r_n^H \rightarrow (\pi/I_0)^2$, as $n \rightarrow \infty$. Thus, $r_n^S = r_n^H \rightarrow r_\infty = (\pi/I_0)^2 \approx 6.8756$. To conclude, the closed-form onset condition derived here demonstrates the occurrence of indefinitely alternating synchronous and subharmonic loop-shaped neutral curves as $\omega \rightarrow 0$. The analytical result serves to close the gap left uncomputed from several numerical studies [4–8] on modulated RB convection. The matched asymptotic WKB method appears promising and is potentially useful for computing other types of low-frequency modulated instabilities.

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